Dynamics of parallel manipulators by means of screw theory

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Abstract

An approach to the dynamic analysis of parallel manipulators is presented. The proposed method, based
on the theory of screws and on the principle of virtual work, allows a straightforward calculation of the
actuator forces as a function of the external applied forces and the imposed trajectory. In order to show the
generality of such a methodology, two case studies are developed, a 2-DOF parallel spherical mechanism
and a Gough–Stewart platform.

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1. Introduction

Over the last two decades parallel manipulators have received increasing attention by kine-
maticians as demonstrated by the relevant amount of published works on this subject, see for
instance [1–9] among many others.

As it is well known a general parallel manipulator is a mechanism composed of a mobile
platform connected to the ground by several independent kinematic chains, called serial connector
chains. Each serial connector chain can be regarded as a serial manipulator with both actuated
and passive joints, the former providing the actuation to the mobile platform. Despite of a re-
duced workspace and a more complex solution of the direct kinematic problem than serial ma-
nipulators, the higher stiffness, accuracy and payload/weight ratio which can be achieved by
parallel manipulators make them attractive systems for applications ranging from motion sim-
ulators to positioning robotic systems. The first suggestion of this class of mechanisms is
historically attributed to Gough, who constructed a prototype of a parallel mechanism for measuring the tire wear and tear under different conditions [2]. Afterwards a similar mechanism was proposed by Stewart, in 1965, as a flight simulator; and nowadays mechanisms that employ the same architecture of the Gough mechanism are without doubt the most studied type of parallel manipulators, universally known in literature as Gough–Stewart platforms.

The dynamic analysis of parallel manipulators has been traditionally carried out through several different methods, i.e. the Newton–Euler method, the Lagrange formulation and the principle of virtual work, which nevertheless present some drawbacks. The Newton–Euler method usually requires large computation time, since it needs the exact calculation of all the internal reactions of constraint of the system, even if they are not employed in the control law of the manipulator. On the other hand, both the Lagrangian and the principle of virtual work formulations are based on the computation of the energy of the whole system with the adoption of a generalized coordinate framework, whereby the system dynamics equations are expressed. Such an energy approach to the analysis of parallel manipulator can be further simplified and standardized by means of the theory of screws.

In this work the analysis of parallel manipulators is developed through a novel methodology based on the theory of screws. The kinematics is approached by extending results previously obtained by the authors [10,12,13], in the analysis of open serial and closed chains to the kinematics of parallel manipulators. Then the dynamics is approached by an harmonious combination of screw theory with the principle of virtual work. Finally, in order to show the effectiveness of the method two applications are given, the first one dealing with a 2-DOF spherical parallel mechanism, and the second one dealing with a Gough–Stewart platform.

2. Preliminary concepts: kinematics of open serial and closed chains

This section summarizes a few results, obtained via screw theory, dealing with the kinematics of open serial and closed chains.

Consider two rigid bodies $i$ and $i+1$, that are connected to each other by means of a helicoidal joint described by the normalized screw $i S^{i+1}$. The screw $i S^{i+1}$ can be expressed in terms of Plücker coordinates as:

$$i S^{i+1} = (i \hat{s}^{i+1}, i \vec{s}^{i+1}),$$

where $i \hat{s}^{i+1}$ is a unit vector defined on the instantaneous rotation axis and $i \vec{s}^{i+1}$ is usually given as a function of the screw pitch $i h_{i+1}$ and the vector $\vec{r}_{O/P}$, directing from an arbitrary point $P$ of the instantaneous screw axis to the point $O$:

$$i \vec{s}^{i+1}_O \equiv i h_{i+1} i \hat{s}^{i+1} + i \vec{s}^{i+1} \times \vec{r}_{O/P},$$

where $\times$ is the usual cross-product of three-dimensional vector algebra.

At the beginning of the last century, Ball [14] defined the velocity state of a rigid body, $\vec{V}_O$, as a screw with a primary component, $\mathcal{P}(\vec{V}) = \vec{o}$, and a dual component, $\mathcal{D}(\vec{V}_O) = \vec{v}_O$, so that

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1 For a historical overview of early analyses on the screw axis, see Ceccarelli [15].
\[ \tilde{V}_O = \begin{bmatrix} \vec{\omega} \\ \vec{v}_O \end{bmatrix}, \] (3)

where \( \vec{\omega} \) is the angular velocity of the considered rigid body and \( \vec{v}_O \) is the velocity of the body point that is coincident with point \( O \). Note that while the primary component does not depend on the point \( O \) and it can be regarded as a property of the particular motion, the dual part depends on the choice of the point \( O \). Hereafter the representation point \( O \) will not be indicated as a subscript of the velocity state. Further, the velocity state of a rigid body can be considered as a twist on a screw, and so the expression (3) can be rewritten as:

\[ \tilde{V}_O = \omega \$. \] (4)

When the representation point is changed from \( O \) to \( P \), only the dual part of the twist \( \tilde{V}_O \) changes as:

\[ \mathcal{D}(\tilde{V}_P) = \mathcal{D}(\tilde{V}) + \mathcal{P}(\tilde{V}) \times \vec{r}_{P/O}. \] (5)

2.1. Kinematics of an open chain manipulator

The direct velocity analysis of a serial manipulator consists of determining the velocity of the end effector when the rates \( \omega_{i} \) of the actuated joints are known. It is well known that in a serial manipulator, see Fig. 1, the velocity state of the end effector, body \( m \), with respect to the base link, body \( 0 \), can be expressed as a linear combination of the screws associated with the serial chain joints:

\[ 0\vec{V}^m = 0\omega_1 0\$.^1 + 1\omega_2 1\$.^2 + \ldots + m-1\omega_m m-1\$.^m, \] (6)

where \( \omega_{i} \) is the joint rate between two consecutive links.\(^2\)

By rearranging Eq. (6) in matrix form, the \( 6 \times m \) Jacobian \( J \) of the manipulator is found as:

\[ 0\vec{V}^m = J \begin{bmatrix} 0\omega_1 \\ 1\omega_2 \\ \vdots \\ m-1\omega_m \end{bmatrix}, \] (7)

where

\[ J = \begin{bmatrix} 0\$.^1 & 1\$.^2 & \ldots & m-1\$.^m \end{bmatrix}. \]

Conversely, the inverse velocity analysis requires the determination of the joint rates \( \omega_{j} \) when the motion of the end effector with respect to the base link is given. If the manipulator is not kinematically redundant and it is not at a singular configuration, the Jacobian is represented by an invertible square matrix, thus:

\(^2\) If the screw \( i\$.^i+1 \) is associated with a prismatic joint, then the joint rate and its screw are \( i\omega_{i+1} i\$.^i+1 = i\tilde{v}_{i+1} \begin{bmatrix} \vec{\omega} \\ \vec{v}_{i+1} \end{bmatrix} \), where \( i\$._{i+1} \) is a unit vector along the direction of the prismatic joint.
The acceleration analysis can be conducted following Brand [16], who defined the acceleration motor \( \tilde{A}_O \) of a rigid body, also known as reduced acceleration state or briefly accelerator, in terms of a primary component, \( \mathcal{P}(\tilde{A}_O) = \tilde{\omega} \), and a dual component, \( \mathcal{D}(\tilde{A}_O) = \tilde{a}_O - \tilde{\omega} \times \tilde{v}_O \):

\[
\tilde{A}_O = \begin{bmatrix}
\tilde{\omega} \\
\tilde{a}_O - \tilde{\omega} \times \tilde{v}_O
\end{bmatrix},
\]

where \( \tilde{\omega} \) and \( \tilde{a}_O \) are, respectively, the angular acceleration of the rigid body and the linear acceleration of a point \( O \) fixed to the rigid body used as a representation point. Further, the acceleration of any other point \( P \) of the rigid body, can be easily computed as:

\[
\tilde{a}_P = \tilde{a}_O + \tilde{\omega} \times \tilde{r}_{P/O} + \tilde{\omega} \times (\tilde{\omega} \times \tilde{r}_{P/O}),
\]

where \( \tilde{r}_{P/O} \) is directed from \( O \) to \( P \). Note that while the primary component of the reduced acceleration state is clearly the angular acceleration, the interpretation of the dual part is not so straightforward. By using concepts of elementary kinematics, it can be easily demonstrated how the dual part of an accelerator is transformed as a screw for a change of the representation point:

\[
\tilde{a}_P = \tilde{a}_O + \tilde{\omega} \times \tilde{r}_{P/O} + \tilde{\omega} \times (\tilde{\omega} \times \tilde{r}_{P/O}) = \tilde{a}_O - \tilde{\omega} \times \tilde{v}_O + \tilde{\omega} \times \tilde{r}_{P/O}.
\]

Due to the difficulty of representing the accelerator in screw form, previous works limited the application of screw theory to velocity analysis only, usually called first order analysis [18].

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3 The velocity and reduced acceleration states can be treated in a unified form by means of the concept of helicoidal vector field [17]. In that way, the velocity analysis can be extended not only to the acceleration analysis but also to the so-called higher order analyses.
Rico et al. [10] showed that the reduced acceleration state of the end effector, body \( m \), of a serial chain can be expressed, in terms of the screws associated to the kinematic pairs of the serial chain, with respect to an inertial reference frame fixed to body 0, as

\[
0 \ddot{A}_m = 0 \dot{\omega}_1 0 \dot{s}_1 + 1 \dot{\omega}_2 1 \dot{s}_2 + \cdots + m-1 \dot{\omega}_{m-1} m-1 \dot{s}_m + s_L,
\]

(12)

where the Lie screw term \( s_L \), is

\[
s_L = [0 \omega_1 0 \dot{s}_1 1 \omega_2 1 \dot{s}_2 + \cdots + m-1 \omega_{m-1} m-1 \dot{s}_m] + \cdots + [m-2 \omega_{m-2} m-2 \dot{s}_{m-2} m-1 \omega_{m-1} m-1 \dot{s}_m],
\]

(13)

and the brackets \([s_1 \ s_2]\) represent the Lie product between the elements \( s_1 \) and \( s_2 \) of the Lie algebra \( e(3) \). By reordering expression (12) in matrix form the direct acceleration analysis can be solved through

\[
0 \ddot{A}_m = J \begin{bmatrix} 0 \dot{\omega}_1 \\ 1 \dot{\omega}_2 \\ \vdots \\ m-1 \dot{\omega}_m \end{bmatrix} + s_L,
\]

(14)

while, under the same conditions of Eq. (8), the inverse acceleration analysis is solved by

\[
\begin{bmatrix} 0 \dot{\omega}_1 \\ 1 \dot{\omega}_2 \\ \vdots \\ m-1 \dot{\omega}_m \end{bmatrix} = J^{-1} (0 \ddot{A}_m - s_L).
\]

(15)

In what follows the results obtained for an open serial chain are extended to closed chains. It is important to note that Eq. (12), and thus Eqs. (14) and (15) are valid under any value of the angular velocity of the end effector, for details the lector is referred to [8,12,17].

2.2. Kinematics of a single closed chain manipulator

A single closed chain manipulator, see Fig. 2, can be obtained by fixing the end effector, body \( m \), of a serial manipulator to the base link, body 0, so that \( 0 \ddot{V}_{m,0} = 0 \ddot{A}_{m,0} = 0 \). Thus, according with Eqs. (6) and (12), the velocity and acceleration analysis of single closed chains must satisfy:
Further, if the resulting closed chain has \( F \) degrees of freedom, with \( F \geq 1 \), then the passive joint rates, \( \dot{\omega}_{ij} \), can be expressed in terms of first order influence coefficients \( \Phi_{ij}^f \), \( i = 1, \ldots, F \), and of generalized velocities \( \dot{q}_i \), as

\[
\dot{\omega}_{ij} = \Phi_{ij}^1 \dot{q}_1 + \Phi_{ij}^2 \dot{q}_2 + \cdots + \Phi_{ij}^F \dot{q}_F, \quad i = 0, \ldots, m-1.
\]

The generic influence coefficient \( \Phi_{ij}^f \), expresses the velocity \( \dot{\omega}_{ij} \) that results at a passive joint \( i \) for a unitary rate of the \( j \)th actuated joint. The computation of the influence coefficients does not present particular difficulty, since for a parallel manipulator it is always possible to write one or more closure equations.

The resulting system of the closure equations is linear in the joint velocities and can be rearranged to express the passive joints rates in terms of the active ones. The influence coefficients for the \( i \)th actuated joint can then be computed by solving the system of equations for \( \dot{q}_i = 1 \) and \( \dot{q}_j = 0 \) with \( j \neq i \). An example of computation is reported in the examples and for further details the reader is referred to Gallardo and Rico [11].

So the velocity state of an intermediate body \( n \) can be computed as

\[
\dot{0}V_n = \dot{0}\omega_1 0\Sigma^1 + \cdots + n-1\omega_1 n^{-1} \Sigma^n,
\]

or substituting Eq. (17), in terms of the first order influence coefficients it follows that

\[
\dot{0}V_n = (\Phi_{10}^1 \dot{q}_1 + \cdots + \Phi_{10}^F \dot{q}_F)0\Sigma^1 + \cdots + (\Phi_{n-10}^1 \dot{q}_1 + \cdots + \Phi_{n-10}^F \dot{q}_F)n^{-1} \Sigma^n
\]

\[
n = 1, \ldots, m-1.
\]

If partial screws \( \Sigma^i_n \), expressing the contribution to the motion of body \( n \) due to the actuated joint \( i \), are defined as:

\[
\Sigma^i_n = \Phi_{i0}^1 0\Sigma^1 + \cdots + \Phi_{i0}^F n^{-1} \Sigma^n \quad i = 1, \ldots, F \quad n = 1, \ldots, m - 1,
\]

thus expression (19), collecting the generalized velocities \( \dot{q}_i \), can be arranged in the following form

\[
\dot{0}V_n = \Sigma^1_n \dot{q}_1 + \cdots + \Sigma^F_n \dot{q}_F \quad n = 1, \ldots, m - 1.
\]

Of course, for numerical computations all the screws must be expressed with respect to the same coordinate system and the same representation point. When the representation point is changed, e.g. from point \( O \) to the center of mass \( C_m \) of body \( n \), the expression of partial screws \( \Sigma^i_n \) in (20) is modified according to (5):

\[
\Sigma^i_{C_n/O} = \left[ \mathcal{P}(\Sigma^i_n) \right] \times \mathcal{P}(\Sigma^i_n). \]

Further, after having expressed the state of motion of a generic body \( n \) through \( \dot{0}V_n \) and \( \dot{0}A_n \), the velocity \( \dot{0}V_n \) and the acceleration \( \dot{0}A_n \) of any arbitrary point of body \( n \), like its center of mass \( C_m \), can be computed respectively by (5) and (10).

Finally, taking into proper account the enumeration of the closed chains, and their links, the results obtained for a single closed chains can be extended to the analysis of parallel manipulators.
3. Kinematics of parallel manipulators

The results obtained in Section 2 for open serial and closed chains can be applied to solve kinematics of parallel manipulators.

Consider a parallel manipulator composed of \( k \) serial connector chains, as shown in Fig. 3, and suppose that the mobile platform has an instantaneous motion \( \overset{\text{O}}{\omega}_{\text{pl}} \) with respect to an inertial reference frame \( O_{\text{XYZ}} \), fixed to the base link, body 0, given by

\[
\overset{\text{O}}{\omega}_{\text{pl}} = \begin{bmatrix}
\overset{\text{O}}{\omega}_{\text{pl}}^1 \\
\overset{\text{O}}{\omega}_{\text{pl}}^2 \\
\vdots \\
\overset{\text{O}}{\omega}_{\text{pl}}^m
\end{bmatrix}.
\] (22)

After expressing all the screws with respect to the point \( O \), the inverse velocity analysis can be individually solved for each serial chain through expression (8):

\[
\begin{bmatrix}
\overset{\text{O}}{\omega}_{1}^{(i)} \\
\overset{\text{O}}{\omega}_{2}^{(i)} \\
\vdots \\
\overset{\text{O}}{\omega}_{m}^{(i)}
\end{bmatrix} = (J^{(i)})^{-1} \overset{\text{O}}{\omega}_{\text{pl}} 
\quad i = 1, \ldots, k,
\] (23)

where

\[
J^{(i)} = \begin{bmatrix}
\overset{\text{O}}{\omega}^{1(i)} & \overset{\text{O}}{\omega}^{2(i)} & \ldots & \overset{\text{O}}{\omega}^{m(i)}
\end{bmatrix}.
\]

Here the superscript \((i)\) denotes the \( i \)th connector chain.

Suppose now that the mobile platform has a reduced acceleration state \( \overset{\text{O}}{a}_{\text{pl}} \) with respect to the inertial reference frame \( O_{\text{XYZ}} \). Then, the inverse acceleration analysis is solved by simple application of expression (15) to each connector chain of the parallel manipulator:

Fig. 3. A parallel manipulator.
The general expressions (23) and (24) are the basis of inverse velocity and acceleration analysis of parallel manipulators. Given a velocity and reduced acceleration state of the moving platform, all the serial connector chains must satisfy Eqs. (23) and (24).

4. Dynamics of parallel mechanisms

Recently the inverse dynamics of a Gough–Stewart platform has been significantly simplified by means of the principle of virtual work [7,9]. In this section it is shown how this approach can be further simplified by applying the Klein form, a bilinear symmetric form of the Lie algebra.

Assume that $\mathbf{S}_1 = (\mathbf{s}_1, \mathbf{s}_{O_1})$ and $\mathbf{S}_2 = (\mathbf{s}_2, \mathbf{s}_{O_2})$ are two elements of the Lie algebra, $e(3)$. The Klein form $KL(\mathbf{S}_1, \mathbf{S}_2)$, a non-degenerate symmetric bilinear form, is defined as

$$KL : e(3) \times e(3) \rightarrow \mathbb{R} \quad KL(\mathbf{S}_1, \mathbf{S}_2) \equiv \mathbf{s}_1 \cdot \mathbf{s}_{O_2} + \mathbf{s}_2 \cdot \mathbf{s}_{O_1},$$

where $\cdot$ denotes the usual dot product of three-dimensional real vector algebra.

The wrench $\mathbf{F}_O$ acting on a rigid body is defined as a screw with a primary component, $\mathbf{P}(\mathbf{F}_O) = \mathbf{f}$, and a dual component, $\mathbf{D}(\mathbf{F}_O) = \mathbf{\omega}$, as follows

$$\mathbf{F}_O = \begin{bmatrix} \mathbf{f} \\ \mathbf{\omega} \end{bmatrix},$$

where $\mathbf{f}$ and $\mathbf{\omega}$ respectively are the force vector and the torque vector expressed by using $O$ as representation point.

Suppose that the velocity and reduced acceleration state, $^0\mathbf{V}_{C_n}^n$ and $^0\mathbf{A}_{C_n}^n$, describe the motion of a rigid body $n$ of mass $m$, adopting its mass center $C_n$ as representation point.

By assuming the same representation point $C_n$, the inertial wrench, $^0\mathbf{F}_{1,C_n}$, acting on body $n$ according with D’Alembert’s principle is given by:

$$^0\mathbf{F}_{1,C_n} = \begin{bmatrix} -m\mathbf{\ddot{c}}_n \\ -I_n\mathbf{\dot{\omega}} - I_n\mathbf{\omega} \times I_n(\mathbf{\omega}) \end{bmatrix},$$

where $I_n = I_n^0$ is the centroidal body inertia matrix expressed in the reference frame $O_{XYZ}$ fixed to the base link. This matrix can be computed by transforming the local inertial matrix $I_n$ through the rotation matrix $^0\mathbf{R}^n$, as

$$I_n = ^0\mathbf{R}^n I_n^0 (^0\mathbf{R}^n)^T.$$
\[
\vec{F}_{G,C_m} = \begin{bmatrix} \vec{mg} \\ 0 \end{bmatrix},
\]

where \( \vec{g} \) is the gravity acceleration vector. Moreover if an external force, \( \vec{f}_E^n \), and an external torque, \( \vec{\tau}_E^n \), are applied to the mass center of the considered rigid body, the external wrench \( \vec{F}_{E,C_m}^n \) is given by:

\[
\vec{F}_{E,C_m}^n = \begin{bmatrix} \vec{f}_E^n \\ \vec{\tau}_E^n \end{bmatrix}.
\]

In the computation of the overall system dynamics, however, the same coordinate system \( O_{XYZ} \) and the same representation point \( C_m \), must be used for the screw and wrench of every link. So Eq. (5) has to be applied in order to express the above written wrenches with respect to a congruent representation pole \( C_m \). Finally, the overall wrench \( \vec{F}_n \) acting on body \( n \) is then

\[
\vec{F}_n = \vec{F}_{1,C_m}^n + \vec{F}_{G,C_m}^n + \vec{F}_{E,C_m}^n.
\]

With these conventions the power \( w_n \), produced by the resulting wrench \( \vec{F}_n \) acting on body \( n \) on a generic motion \( ^0\vec{V}_m^n \), can be determined with the aid of the Klein form, Eq. (25), as follows

\[
w_n = KL(\vec{F}_n, ^0\vec{V}_m^n).
\]

If the body \( n \) belongs to a closed chain, with \( m - 1 \) (intermediate) bodies, by substituting Eq. (20) into Eq. (32), the total power \( w \) performed on the mechanism by the external forces and actuated joints acting on all links is found as:

\[
w = KL(\vec{F}_1, ^1s_{C_m}^1\dot{q}_1 + \cdots + ^Fs_{C_m}^1\dot{q}_F) + KL(\vec{F}_2, ^1s_{C_m}^2\dot{q}_1 + \cdots + ^Fs_{C_m}^2\dot{q}_F) + \cdots \\
+ KL(\vec{F}_{m-1}, ^{m-1}s_{C_m}^m\dot{q}_1 + \cdots + ^Fs_{C_m}^m\dot{q}_F) + \tau_1\dot{q}_1 + \cdots + \tau_F\dot{q}_F,
\]

where \( \tau_i \) is the generalized force actuating on the \( i \)th joint with the generalized velocity \( \dot{q}_i \). So collecting the generalized velocities \( \dot{q}_i \):

\[
w = [KL(\vec{F}_1, ^1s_{C_m}^1) + KL(\vec{F}_2, ^1s_{C_m}^2) + \cdots + KL(\vec{F}_{m-1}, ^1s_{C_m}^{m-1}) + \tau_1]\dot{q}_1 + \cdots \\
+ [KL(\vec{F}_1, ^1s_F^1) + KL(\vec{F}_2, ^1s_F^2) + \cdots + KL(\vec{F}_{m-1}, ^1s_F^{m-1}) + \tau_F]\dot{q}_F.
\]

The principle of virtual work states that if a multi-body system is in equilibrium under the effect of external actions, then the global work produced by the external forces with any virtual velocity must be zero, for details the reader is referred to Tsai [8]. Introducing the generalized virtual velocities \( \delta \dot{q}_i \), and taking into account that the work due to the internal reactions of constraint is zero, the global virtual work \( \delta w \), from (34), is given by:

\[
\delta w = [KL(\vec{F}_1, ^1s_{C_m}^1) + KL(\vec{F}_2, ^1s_{C_m}^2) + \cdots + KL(\vec{F}_{m-1}, ^1s_{C_m}^{m-1}) + \tau_1]\delta \dot{q}_1 + \cdots \\
+ [KL(\vec{F}_1, ^1s_F^1) + KL(\vec{F}_2, ^1s_F^2) + \cdots + KL(\vec{F}_{m-1}, ^1s_F^{m-1}) + \tau_F]\delta \dot{q}_F.
\]

Further, since the generalized virtual velocities \( \delta \dot{q}_i \) are arbitrary, \( \delta w = 0 \) if and only if:

\[
KL(\vec{F}_1, ^1s_{C_m}^j) + KL(\vec{F}_2, ^1s_{C_m}^j) + \cdots + KL(\vec{F}_{m-1}, ^1s_{C_m}^{j-1}) + \tau_j = 0 \quad i = 1, \ldots, F
\]
and so
\[
\tau_i = -\{\vec{f}^1 \cdot \vec{s}_{C_{m1}} + \vec{f}^2 \cdot \vec{s}_{C_{m2}} + \vec{f}^3 \cdot \vec{s}_{C_{m}} + \vec{f}^{m-1} \cdot \vec{s}_{C_{m-1}} + \vec{f}^{m} \cdot \vec{s}_{C_{m-1}}\} \quad i = 1, \ldots, F.
\]

(37)

Finally, from expression (37) the generalized forces \(\tau_i\) can be computed directly. It is important to note that from the above solution, \(\tau_i\) are independent from the virtual velocities \(\delta \dot{q}_i\), but they obviously still depend on the real generalized velocities \(\dot{q}_i\), contained in the inertial terms of (37), through the Lie bracket expression contained in the computation of the links accelerations.

5. Example 1, 2-DOF spherical mechanism

This section shows a first application of the method to a 2-DOF spatial parallel mechanism used as a haptic interface to simulate a virtual car gearshift. The calculation of the system dynamics will be approached by solving the direct kinematic equation of the system. The kinematic representation of the spherical mechanism is shown in Fig. 4 together with the ground coordinate system \(O_{XYZ}\) and the joint axes.

The differential kinematic of the mechanism is easily derived using screw algebra. Since the system is spherical, the fixed point \(O\) is conveniently chosen as the representation point.

Taking into account that all the axes of the screws are concurrent and noting that the axes of screws \(1^2(2)\) and \(1^2(1)\) are always orthogonal to the knob \(OP\), then in Plücker coordinates the infinitesimal screws are given by

\[
\begin{align*}
0^1(1) &= (1, 0, 0; \vec{0}) \\
0^1(2) &= (0, 1, 0; \vec{0}) \\
1^2(1) &= (0, \cos(q_1), \sin(q_1); \vec{0}) \\
1^2(2) &= (\cos(q_2), 0, -\sin(q_2); \vec{0}),
\end{align*}
\]

(38)

\[
\begin{align*}
0^3(1) &= (2, \cos(q_1), \sin(q_1); \vec{0}) \\
0^3(2) &= (\cos(q_2), 0, -\sin(q_2); \vec{0}).
\end{align*}
\]

(39)

Fig. 4. The gearshift mechanism and its scheme.
where

\[
\begin{align*}
\hat{q}(1) &= \frac{1}{a}((1) \times 1) = \frac{1}{a}(\cos(q_1) \sin(q_2), -\sin(q_1) \cos(q_2), \cos(q_1) \cos(q_2)), \\
a &= \|1 \times 1\|,
\end{align*}
\]

where, given a position of the free end of the knob by \(\vec{P} = (P_x, P_y, P_z)\), the generalized coordinates \(q_1\) and \(q_2\) can be obtained as

\[
q_1 = \arctan(-P_y/P_x) \quad q_2 = \arctan(P_x/P_y).
\]

Assume that a velocity state of the knob with respect to the base link, \(0\vec{V}^3(1) = (0\vec{\omega}^3(1), \vec{0})\), is required. Then, following the two connector chains, it is possible to write

\[
0\vec{V}^3(1) = \dot{q}_2 0 \hat{s}^1(2) + \omega_1 1 \hat{s}^2(2) = \dot{q}_1 0 \hat{s}^1(1) + \omega_2 1 \hat{s}^2(1) + 2 \omega_3 2 \hat{s}^3(1).
\]

Thus, after cancelling the dual parts of the infinitesimal screws, the joint rates can be computed. Indeed

\[
\begin{bmatrix}
\mathcal{P}^0 \hat{s}^1(2) \\
\mathcal{P}^1 \hat{s}^2(2) \\
0
\end{bmatrix}
\begin{bmatrix}
\dot{q}_2 \\
\omega_2(2) \\
0
\end{bmatrix}
= 0 \hat{\omega}^3(1),
\]

\[
\begin{bmatrix}
\dot{q}_1 \\
\omega_2(1) \\
2 \omega_3(1)
\end{bmatrix}
= \left[ \begin{bmatrix}
\mathcal{P}^0 \hat{s}^1(1) \\
\mathcal{P}^1 \hat{s}^2(1) \\
\mathcal{P}^2 \hat{s}^3(1)
\end{bmatrix} \right]^{-1} 0 \hat{\omega}^3(1).
\]

Further, from expression (40) it follows that

\[
\begin{bmatrix}
1 \hat{s}^2(1) \\
2 \hat{s}^3(1) \\
-1 \hat{s}^2(2)
\end{bmatrix}
\begin{bmatrix}
\omega_2(1) \\
2 \omega_3(1) \\
\omega_2(2)
\end{bmatrix}
= -\dot{q}_1 0 \hat{s}^1(1) + \dot{q}_2 0 \hat{s}^1(2).
\]

Afterwards, the influence coefficients are calculated by simple application of Cramer’s rule. Only the influence coefficients associated to the joint rate \(1\omega_2(1)\) will be presented here.

From expression (42) immediately emerges that:

\[
1\omega_2(1) = \frac{1}{\left| \begin{bmatrix}
0 \hat{s}^1(1) & 2 \hat{s}^3(1) & 1 \hat{s}^2(2)
\end{bmatrix}
\end{bmatrix}} \begin{bmatrix}
-\dot{q}_1 0 \hat{s}^1(1) + \dot{q}_2 0 \hat{s}^1(2) \\
2 \hat{s}^3(1) \\
1 \hat{s}^2(2)
\end{bmatrix}.
\]

Thus,

\[
1\omega_2(1) = G^1_{1\omega_2(1)} \dot{q}_1 + G^2_{1\omega_2(1)} \dot{q}_2,
\]

where the first order coefficients are given by

\[
G^1_{1\omega_2(1)} = \frac{1}{\left| \begin{bmatrix}
0 \hat{s}^1(1) & 2 \hat{s}^3(1) & 1 \hat{s}^2(2)
\end{bmatrix}
\end{bmatrix}} = -\sin(q_1) \sin(q_2) \cos(q_2) / a^2,
\]

\[
G^2_{1\omega_2(1)} = \frac{1}{\left| \begin{bmatrix}
0 \hat{s}^1(2) & 2 \hat{s}^3(1) & 1 \hat{s}^2(2)
\end{bmatrix}
\end{bmatrix}} = \cos(q_1) / a^2.
\]
Similarly by application of Cramer’s rule it is found:

\[
G_{20^{(1)}}^1 = -\frac{1\dot{s}_2^{(2)} 0\dot{s}_1^{(1)} 1\dot{s}_2^{(2)}}{1\dot{s}_3^{(1)} 2\dot{s}_3^{(1)} 1\dot{s}_2^{(2)}} = \frac{-\cos(q_1) \sin(q_2)}{a},
\]

\[
G_{20^{(1)}}^2 = \frac{1\dot{s}_2^{(2)} 0\dot{s}_1^{(2)} 1\dot{s}_2^{(2)}}{1\dot{s}_3^{(1)} 2\dot{s}_3^{(1)} 1\dot{s}_2^{(2)}} = \frac{-\sin(q_1) \cos(q_2)}{a}.
\]

Thus, the partial screws associated to the knob (body 3 of chain 1), in terms of influence coefficients are given by

\[
\begin{align*}
\mathbf{s}_{3}^{1} &= 0\mathbf{s}_{1}^{1} + G_{10_{2}^{(1)}}^{1} \mathbf{s}_{3}^{1} + G_{20_{3}^{(1)}}^{1} 2\mathbf{s}_{3}^{1}, \\
\mathbf{s}_{3}^{2} &= G_{10_{2}^{(1)}}^{2} \mathbf{s}_{2}^{1} + G_{20_{3}^{(1)}}^{2} 2\mathbf{s}_{3}^{1}.
\end{align*}
\]

Finally, the corresponding partial screws of the knob relative to its mass center, \(C_{m3(1)}\), expressed in the global reference frame \(O_{XYZ}\), result in

\[
\mathbf{s}_{Cm3(1)}^{i} = \left[ \mathcal{P}(\mathbf{s}_{3}^{i}) \times \mathbf{n}_{Cm3(1)}/O \right] \\
\text{where } i = 1, 2, \quad (47)
\]

\[\text{here } \mathbf{n}_{Cm3(1)/O} \text{ is the vector pointed from the common intersection of all the revolute axes, point } O, \text{ to the mass center of the knob.}\]

It is straightforward to demonstrate that the partial screws \(\mathbf{s}_{Cm3(1)}^{i}\) of the remaining bodies are obtained in a similar way.

Assume that a reduced acceleration state of the knob with respect to the base link, \(0\mathbf{\ddot{A}}^{3(1)} = (0\mathbf{\ddot{\omega}}_{3(1)}; \mathbf{0})\), is required. Then, according with the two connector chains, it is possible to write

\[
0\mathbf{\ddot{A}}^{3(1)} = \mathbf{\ddot{q}}_2 0\mathbf{s}_1^{1(2)} + 1\mathbf{\ddot{\omega}}_{2(1)} 1\mathbf{s}_2^{1(2)} + \mathbf{\ddot{\omega}}_{1(1)} = \mathbf{\ddot{q}}_1 0\mathbf{s}_1^{1(1)} + 1\mathbf{\ddot{\omega}}_{2(1)} 1\mathbf{s}_2^{1(1)} + 2\mathbf{\ddot{\omega}}_{3(1)} 2\mathbf{s}_3^{1(1)} + \mathbf{s}_{L2}, \quad (48)
\]

where

\[
\begin{align*}
\mathbf{s}_{L1} &= \left[ \mathbf{\ddot{q}}_2 0\mathbf{s}_1^{1(2)} \right] + \left[ \mathbf{\ddot{\omega}}_{2(1)} 1\mathbf{s}_2^{1(2)} \right], \\
\mathbf{s}_{L2} &= \left[ \mathbf{\ddot{q}}_1 0\mathbf{s}_1^{1(1)} \right] + \left[ \mathbf{\ddot{\omega}}_{2(1)} 1\mathbf{s}_2^{1(1)} + 2\mathbf{\ddot{\omega}}_{3(1)} 2\mathbf{s}_3^{1(1)} \right] + \left[ 1\mathbf{\ddot{\omega}}_{2(1)} 1\mathbf{s}_2^{1(1)} + 2\mathbf{\ddot{\omega}}_{3(1)} 2\mathbf{s}_3^{1(1)} \right].
\end{align*}
\]

After cancelling the dual part of all the six dimensional vectors, the joint acceleration rates can be computed from expression (48). Indeed

\[
\begin{bmatrix}
\mathcal{P}(0\mathbf{s}_1^{1(2)}) & \mathcal{P}(1\mathbf{s}_2^{1(2)}) & \mathbf{0} \\
\mathbf{0} & \mathcal{P}(1\mathbf{s}_2^{1(2)}) & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathcal{P}(1\mathbf{s}_2^{1(2)})
\end{bmatrix}
\begin{bmatrix}
\mathbf{\ddot{q}}_1 \\
\mathbf{\ddot{\omega}}_{2(1)} \\
\mathbf{\ddot{\omega}}_{3(1)}
\end{bmatrix}
= (0\mathbf{\ddot{\omega}}_{3(1)} - \mathcal{P}(\mathbf{s}_{L1})),
\]

\[
\begin{bmatrix}
\mathbf{\ddot{q}}_1 \\
\mathbf{\ddot{\omega}}_{2(1)} \\
\mathbf{\ddot{\omega}}_{3(1)}
\end{bmatrix}
= \left[ \mathcal{P}(0\mathbf{s}_1^{1(1)}) \mathcal{P}(1\mathbf{s}_2^{1(1)}) \mathcal{P}(2\mathbf{s}_3^{1(1)}) \right]^{-1} (0\mathbf{\ddot{\omega}}_{3(1)} - \mathcal{P}(\mathbf{s}_{L2})).
\]

Once the inverse analyses are completed, the velocity and the acceleration of the mass center of the knob are computed by application of elementary kinematics.
Consider now a frame $O_{i,j}^{0}$ attached to each link, used to locally describe its inertial properties. If we consider local frames whose origin $O_{i,j}$ is coincident with that of the base coordinate system $O^{0}$ the transformation matrices reduce to rotation matrices. Defining the frame $O_{i,j}^{0}$ of the link $i,j$ according to the Denavit–Hartenberg notation, we can quickly build the rotation matrix $R_{i,j}^{0}$ that transforms local properties in the ground frame $O^{0}$.

If $r_{i,j}^{*}$ is the vector, in local coordinates, directing from the origin $O$ to the center of gravity $B_{i,j}$ of the link-$i,j$, in the ground frame it becomes $r_{i,j}^{*} = R_{i,j}^{0}r_{i,j}^{*}$. The inertia tensor in ground frame is expressed by:

$$I_{0} = R_{i,j}^{0}I_{0}^{*}(R_{i,j}^{0})^{t}.$$ (51)

Consider now a generic link, and express all the forces acting on it with respect to ground frame. As the system is characterized by a spherical kinematics, only the momentum part of each applied wrench must be considered for calculating the produced virtual work. The gravity force acting on each link produces the momentum:

$$\vec{M}_{g,i(j)} = R_{i,j}^{0}r_{B_{i,j}/O}^{*} \times m_{i(j)}\vec{g}.$$ (52)

If $\vec{\omega}_{L_{i,j}}$ and $\vec{\alpha}_{L_{i,j}}$ are the $i$-link angular velocity and acceleration of serial connector chain $j$, then the momentum of the inertial forces is:

$$\vec{M}_{L_{i(j)}} = -I_{0}\vec{\omega}_{L_{i(j)}} - \vec{\alpha}_{L_{i(j)}} \times I_{0}\vec{\omega}_{L_{i(j)}}.$$ (53)

Indicating with $\vec{M}_{e}$ the external torque, applied to the handle by an external force $\vec{F}_{e}$, we can now compute all the torques acting on the mechanism. Therefore, the wrenches are given by

$$\begin{align*}
F^{1(1)} &= \begin{bmatrix}
m_{1(1)}\vec{g} \\
\vec{M}_{g,1(1)} + \vec{M}_{1(1)}
\end{bmatrix}, \\
F^{2(1)} &= \begin{bmatrix}
m_{2(1)}\vec{g} \\
\vec{M}_{g,2(1)} + \vec{M}_{1(2)} + \vec{M}_{e}
\end{bmatrix}, \\
F^{3(1)} &= \begin{bmatrix}
m_{3(1)}\vec{g} + \vec{F}_{e} \\
\vec{M}_{g,2(1)} + \vec{M}_{1(2)} + \vec{M}_{e}
\end{bmatrix}, \\
F^{1(2)} &= \begin{bmatrix}
m_{2(2)}\vec{g} \\
\vec{M}_{g,2(1)} + \vec{M}_{1(2)} + \vec{M}_{e}
\end{bmatrix}.
\end{align*}$$

Finally, the expression of application (37) leads to

$$\begin{align*}
KL(F^{1(1)}, S_{C_{1}(1)}^{1}) &+ KL(F^{2(1)}, S_{C_{1}(2)}^{1}) + KL(F^{3(1)}, S_{C_{1}(3)}^{1}) + KL(F^{1(2)}, S_{C_{2}(1)}^{1}) + \tau_{1} = 0, \\
KL(F^{1(1)}, S_{C_{1}(1)}^{2}) &+ KL(F^{2(1)}, S_{C_{1}(2)}^{2}) + KL(F^{3(1)}, S_{C_{1}(3)}^{2}) + KL(F^{1(2)}, S_{C_{2}(1)}^{2}) + \tau_{2} = 0.
\end{align*}$$ (54)

From this last expression, the generalized forces $\tau_{1}$ and $\tau_{2}$ can be computed directly:

$$\begin{align*}
\tau_{1} &= -[m_{1(1)}\vec{g} \cdot \vec{s}_{C_{1}(1)}^{1} + (\vec{M}_{g,1(1)} + \vec{M}_{1(1)}) \cdot \vec{s}_{C_{1}(1)}^{1} + m_{2(1)}\vec{g} \cdot \vec{s}_{C_{2}(1)}^{1}] \\
&\quad + (\vec{M}_{g,2(1)} + \vec{M}_{1(2)}) \cdot \vec{s}_{C_{1}(1)}^{2} + (m_{3(1)}\vec{g} + \vec{F}_{e}) \cdot \vec{s}_{C_{1}(3)}^{1} \\
&\quad + (\vec{M}_{g,2(1)} + \vec{M}_{1(2)} + \vec{M}_{e}) \cdot \vec{s}_{C_{1}(1)}^{1} + m_{1(2)}\vec{g} \cdot \vec{s}_{C_{2}(1)}^{1} + (\vec{M}_{g,1(2)} + \vec{M}_{1(2)}) \cdot \vec{s}_{C_{1}(1)}^{2}], \\
\tau_{2} &= -[m_{1(1)}\vec{g} \cdot \vec{s}_{C_{1}(1)}^{2} + (\vec{M}_{g,1(1)} + \vec{M}_{1(1)}) \cdot \vec{s}_{C_{1}(1)}^{2} + m_{2(1)}\vec{g} \cdot \vec{s}_{C_{2}(1)}^{1}] \\
&\quad + (\vec{M}_{g,2(1)} + \vec{M}_{1(2)}) \cdot \vec{s}_{C_{1}(1)}^{2} + (m_{3(1)}\vec{g} + \vec{F}_{e}) \cdot \vec{s}_{C_{1}(3)}^{1} \\
&\quad + (\vec{M}_{g,2(1)} + \vec{M}_{1(2)} + \vec{M}_{e}) \cdot \vec{s}_{C_{1}(1)}^{2} + m_{1(2)}\vec{g} \cdot \vec{s}_{C_{2}(1)}^{2} + (\vec{M}_{g,1(2)} + \vec{M}_{1(2)}) \cdot \vec{s}_{C_{1}(1)}^{2}].
\end{align*}$$ (55)
6. Example 2, Gough–Stewart manipulator

This section presents an application of the presented method to the analysis of a Gough–Stewart manipulator Fig. 5.

The mobile platform is connected to the base link by means of six serial chains. Each serial connector chain is composed of a lower spherical joint, a prismatic actuated joint and an upper universal joint. The prismatic actuated joints provide six degrees of freedom to the mobile platform, so that it can assume an arbitrary pose. The actuators lengths are defined by 12 points, namely \( U^{(i)} \) and \( S^{(i)} \), respectively fixed to the mobile platform and to the base link. At once, each serial connector chain is composed of a lower link, \( l^{(i)} \), and an upper link, \( u^{(i)} \).

6.1. Position analysis

With respect to the direct kinematics problem, the inverse position analysis of parallel manipulators is quite simple and can be solved by using homogeneous transformation matrices, \( T_{4 \times 4} \).

Consider two reference frames, the first one, \( o_{xyz} \), attached to the mobile platform, and the second one, \( O_{XYZ} \), attached to the base link. The coordinates of a point fixed to the mobile platform, expressed in the reference frame \( o_{xyz} \), can be expressed in the reference frame \( O_{XYZ} \) as follows

\[
\begin{bmatrix}
X \\
Y \\
Z \\
1
\end{bmatrix} = 0^T_{\text{pl}} \begin{bmatrix}
x \\
y \\
z \\
1
\end{bmatrix},
\]

where the transformation matrix \( 0^T_{\text{pl}} \) is given by the rotation matrix \( R \) and the translation vector \( \vec{r}_{o/O} \), that points to the origin of the reference system \( o_{xyz} \) from the origin of the reference system \( O_{XYZ} \):

\[
0^T_{\text{pl}} = \begin{bmatrix}
R & \vec{r}_{o/O} \\
0_{1 \times 3} & 1
\end{bmatrix}.
\]

Fig. 5. A Gough–Stewart manipulator.
Thus the instantaneous position of point \( U^{(i)} = (U_X^{(i)}, U_Y^{(i)}, U_Z^{(i)}) \), on the mobile platform, with respect to the reference system \( O_{XYZ} \) is given by

\[
\begin{bmatrix}
U_X^{(i)} \\
U_Y^{(i)} \\
U_Z^{(i)} \\
1
\end{bmatrix} =^0 T^{pl}_{i}
\begin{bmatrix}
U_X^{(i)} \\
U_Y^{(i)} \\
U_Z^{(i)} \\
1
\end{bmatrix} \quad i = 1, \ldots, 6.
\tag{58}
\]

Once the inverse position analysis is solved, the length of each actuator, connecting the mobile platform to the base link can be determined as the distance between points \( U^{(i)} \) and \( S^{(i)} \), where \( i = 1, \ldots, 6 \).

### 6.2. Velocity analysis

The computation of the generalized velocities \( \dot{q}_i \), a necessary step in the computation of the influence coefficients, can be shortened by applying the properties of the Klein form, for details the reader is referred to Rico and Duffy [5].

Assume that the platform has a velocity state, with respect to the base link, \( ^0 \dot{V}^{pl} \), then according with expression (7), it is possible to write

\[
0 \dot{V}^{pl} = ^0 \dot{V}^{m(i)} = J^{(i)}
\begin{bmatrix}
0 \omega_1^{(i)} \\
1 \omega_2^{(i)} \\
2 \omega_3^{(i)} \\
3 \omega_4^{(i)} \\
4 \omega_5^{(i)} \\
5 \omega_6^{(i)}
\end{bmatrix} \quad i = 1, 2, \ldots, 6,
\tag{59}
\]

where

\[
J^{(i)} =
\begin{bmatrix}
0 s_1^{(i)} & 1 s_2^{(i)} & 2 s_3^{(i)} & 3 s_4^{(i)} & 4 s_5^{(i)} & 5 s_6^{(i)}
\end{bmatrix}.
\]

Note that the screw sets \( \{0 s_1^{(i)}, 1 s_2^{(i)}, 2 s_3^{(i)}\} \) and \( \{4 s_5^{(i)}, 5 s_6^{(i)}\} \) represent, respectively, the lower spherical pair \( S^{(i)} \) and the upper universal joint \( U^{(i)} \), while the screw \( 3 s_4^{(i)} \) is associated to the actuator joint.

Introduce a unit line, \( s^{(i)} = (\hat{s}^{(i)}, \hat{s}^{(i)}_O) \), whose direction is given by \( \varphi(3 s_4^{(i)}) \) and passes through \( U^{(i)} \) and \( S^{(i)} \). It is evident that this line is reciprocal to all the screws representing the revolute pairs of the \( i \)th serial connector chain. Thus, applying the Klein form to both sides of expression (59) it follows that \( KL(\hat{s}^{(i)}, 0 \hat{V}^{pl}) = 0 \) for \( j = 0, 1, 2, 4, 5 \) and therefore

\[
\dot{q}_i = KL(\hat{s}^{(i)}, ^0 \dot{V}^{pl}) = \hat{s}^{(i)T} A^0 \dot{V}^{pl} \quad i = 1, 2, \ldots, 6,
\tag{60}
\]

where

\[
A =
\begin{bmatrix}
0_{3x3} & I_3 \\
I_3 & 0_{3x3}
\end{bmatrix} = A^{-1}.
\]

Thus, it is possible to write in abbreviated form
In what follows, the partial screws of the lower links will be computed. The velocity state of the mobile platform, $\dot{\mathbf{q}}$, is given by

$$\mathbf{J}_q^T \Delta (\dot{\mathbf{q}})^T \mathbf{V}^\text{pl} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix},$$

(61)

where

$$\mathbf{J}_q = \begin{bmatrix} \mathbf{J}_1^{(1)} \\ \mathbf{J}_2^{(2)} \\ \mathbf{J}_3^{(3)} \\ \mathbf{J}_4^{(4)} \\ \mathbf{J}_5^{(5)} \\ \mathbf{J}_6^{(6)} \end{bmatrix}.$$  

With expression (61) the generalized velocities can be quickly computed. Thus, the partial screws of the mobile platform, $\mathbf{S}^i_{\text{pl}}$, are given by

$$\mathbf{S}^i_{\text{pl}} = (\mathbf{J}_q^T \Delta)^{-1} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} j = 1, 2, \ldots, 6.$$  

(62)

Further, the substitution of expression (59) into expression (61) leads to

$$\mathbf{J}_q^T \Delta \dot{\mathbf{q}}^{(i)} = \begin{bmatrix} 0 \mathbf{O}_1^{(i)} \\ \mathbf{1}_{\mathbf{O}_2^{(i)}} \\ 2 \mathbf{O}_3^{(i)} \\ 3 \mathbf{O}_4^{(i)} \\ 4 \mathbf{O}_5^{(i)} \\ 5 \mathbf{O}_6^{(i)} \end{bmatrix} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \end{bmatrix} i = 1, 2, \ldots, 6.$$  

(63)

Expression (63) allows to determine, via Cramer’s rule, the influence coefficients, $G_{i,\omega_{j+1}}$, for $j = 1, \ldots, F$, associated to each one of the joint rates. Thus, the joint rates can be computed as

$$\dot{\mathbf{O}}_{i,\omega_{j+1}}^{(j)} = G_{i,\omega_{j+1}}^{(j)} \dot{q}_1 + G_{i,\omega_{j+1}}^{(j)} \dot{q}_2 + \cdots + G_{i,\omega_{j+1}}^{(j)} \dot{q}_6 i = 0, \ldots, 5 \quad j = 1, 2, \ldots, 6.$$  

(64)

In what follows, the partial screws of the lower links will be computed. The velocity state of the $j$th lower link, see expression (19), is given by

$$0 \mathbf{V}^{3(j)} = (G_1^{(1)} \dot{q}_1 + \cdots + G_6^{(1)} \dot{q}_6) \mathbf{S}^{1(j)} + (G_1^{(2)} \dot{q}_1 + \cdots + G_6^{(2)} \dot{q}_6) \mathbf{S}^{2(j)} + \cdots + (G_1^{(6)} \dot{q}_1 + \cdots + G_6^{(6)} \dot{q}_6) \mathbf{S}^{6(j)}$$

$$+ (G_1^{(1)} \dot{q}_1 + \cdots + G_6^{(1)} \dot{q}_6) \mathbf{S}^{3(j)} j = 1, 2, \ldots, 6.$$  

(65)

Thus

$$0 \mathbf{V}^{3(j)} = \mathbf{S}^{1(j)} \dot{q}_1 + \mathbf{S}^{2(j)} \dot{q}_2 + \mathbf{S}^{3(j)} \dot{q}_3 + \mathbf{S}^{4(j)} \dot{q}_4 + \mathbf{S}^{5(j)} \dot{q}_5 + \mathbf{S}^{6(j)} \dot{q}_6,$$

(66)

where the partial screws are given by

$$\mathbf{S}^{i(j)} = G_{i,\omega_{j+1}}^{(1)} \mathbf{S}^{1(j)} + G_{i,\omega_{j+1}}^{(2)} \mathbf{S}^{2(j)} + \cdots + G_{i,\omega_{j+1}}^{(6)} \mathbf{S}^{6(j)} i = 1, 2, \ldots, 6.$$  

(67)
Finally, the partial screws relative to the mass center of the $i$th lower link result in

$$
\mathbf{s}_{L}^{i} = \left[ \mathbf{P}(\mathbf{s}_{0}^{i}) + \mathbf{P}(\mathbf{s}_{1}^{i}) \times \mathbf{r}_{0}^{i} \right] \quad i = 1, 2, \ldots, 6,
$$

(68)

where $\mathbf{r}_{0}^{i}$ is a vector pointed from $\mathbf{s}_{0}^{i}$ to the mass center of the lower link. It is straightforward to demonstrate that the partial screws of the upper links can be determined analogously.

### 6.3. Acceleration analysis

Given a reduced acceleration state of the platform with respect to the base link, $\mathbf{A}_{pl}$, by applying the expression (24), the inverse acceleration analysis of each connector chain is determined by:

$$
\begin{bmatrix}
0 \omega_{1}^{(i)} \\
1 \omega_{2}^{(i)} \\
2 \omega_{3}^{(i)} \\
\ddot{q}_{i} \\
4 \omega_{4}^{(i)} \\
5 \omega_{5}^{(i)}
\end{bmatrix}
= (\mathbf{J}^{(i)})^{-1}(\mathbf{A}_{pl} - \mathbf{s}_{L}^{(i)})
\quad i = 1, 2, \ldots, 6,
$$

(69)

where

$$
\mathbf{s}_{L}^{(i)} = \left[ 0 \omega_{1}^{(i)} \frac{\mathbf{g}}{2}, 1 \omega_{2}^{(i)} \frac{\mathbf{g}}{2}, 2 \omega_{3}^{(i)} \frac{\mathbf{g}}{2}, \ddot{q}_{i} \frac{\mathbf{g}}{2}, 4 \omega_{4}^{(i)} \frac{\mathbf{g}}{2}, 5 \omega_{5}^{(i)} \frac{\mathbf{g}}{2} \right]
$$

(70)

that is a homogeneous linear system with six equations in the six unknowns,

$$
\{0 \omega_{1}^{(i)}, 1 \omega_{2}^{(i)}, 2 \omega_{3}^{(i)}, \ddot{q}_{i}, 4 \omega_{4}^{(i)}, 5 \omega_{5}^{(i)}\}.
$$

When the velocity and acceleration analyses are completed, the velocity and the acceleration of the mass center of each body are computed by application of elementary kinematics.

### 6.4. Computation of generalized forces

Supposing that an external force $\mathbf{f}_{E}^{pl}$ and torque $\mathbf{\tau}_{E}^{pl}$ are applied to the mass center of the mobile platform, then the wrench resulting, see expression (31), on the platform is given by

$$
\mathbf{F}_{cm}^{pl} = \begin{bmatrix}
m_{pl}(\mathbf{g} - \ddot{c}_{cm}) + f_{E}^{pl} \\
\mathbf{\tau}_{E}^{pl} - l_{0}^{pl} \omega_{pl} - \dot{\omega}_{pl} \times l_{0}^{pl} \omega_{pl}
\end{bmatrix}.
$$

(71)

Further, supposing, as a practical assumption, that there are no external forces applied to the chain bodies, then the wrench resulting on the lower links of the manipulator are given by
\[ \vec{F}^{(i)} = \begin{bmatrix} m \vec{l}^{(i)}(\vec{g} - \vec{a}^{(i)}) \\ -I_0^{(i)} \vec{\omega}^{(i)} - \dot{\vec{\omega}}^{(i)} \times I_0^{pl} \vec{\omega}^{(i)} \end{bmatrix} i = 1, 2, \ldots, 6. \]  \tag{72}

With regards for the upper links

\[ \vec{F}^{u(i)} = \begin{bmatrix} m \vec{l}^{(i)}(\vec{g} - \vec{a}^{(i)}) \\ -I_0^{(i)} \vec{\omega}^{(i)} - \dot{\vec{\omega}}^{(i)} \times I_0^{pl} \vec{\omega}^{(i)} \end{bmatrix} i = 1, 2, \ldots, 6, \]  \tag{73}

where \( \vec{a}^{(i)} \) and \( \vec{a}^{u(i)} \) are the accelerations of the mass center of the lower and upper links respectively.

Finally, applying expression (37) it follows that

\[
KL(\vec{F}^{u(1)}; \vec{s}_{uCm(1)}^l) + \cdots + KL(\vec{F}^{u(6)}; \vec{s}_{uCm(6)}^l) + KL(\vec{F}^{l(1)}; \vec{s}_{lCm(1)}^l) + \cdots + KL(\vec{F}^{l(6)}; \vec{s}_{lCm(6)}^l) + KL(\vec{F}^{pl}; \vec{s}_{pl}^l) + \tau_i = 0 \quad i = 1, \ldots, 6. \]  \tag{74}

Thus, the generalized forces \( \tau_i \) can be computed directly from expression (74).

### 7. Conclusions

This paper shows how the screw theory and the principle of virtual work can be used to systematically solve the dynamics of parallel manipulators. Unlike other procedures reported in the literature, the proposed method does not require the computation of internal forces of constraint, which is an unnecessary task when the goal of the analysis is neither the dimensioning of the elements nor the computation of the kinetic energy of the whole system. Two examples were analyzed, and the general expressions so obtained for the complete analysis show the versatility of the presented method.

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### References


