Hamiltonian Formulation of the Constrained Dynamics of a Tendon Driven Parallel Mechanism

Antonio Frisoli and Massimo Bergamasco

PERCRO, Scuola Superiore S. Anna, Pisa-Italy

Abstract This paper presents an Hamiltonian formulation of the constrained dynamics of a novel tendon driven parallel mechanism. The mechanism is a 2-dof closed 5-bar linkage actuated through a special tendon drive which enhances the kinematic isotropy of the system. The dynamic equations of the system are derived considering the additional degrees of freedom generated by the compliance of the tendon drive. The linearizable and zero-dynamics of the system are shown.

1 Introduction

Parallel mechanisms present several well-known advantages in terms of achievable performance with respect to serial ones. With particular regard to planar mechanisms, either different performance measures or optimization methods have been proposed for the closed 5-bar linkage, both from a kinematic (Stocco, 1998; Gao, 1998; Chung, 2001) and a dynamic (Hayward, 1994) perspective. Frisoli (1999) showed that the adoption of a tendon drive for the actuation of a parallel mechanism represents a method to enhance its kinematics and dynamics global isotropy. On this principle, a mechanical system based on a tendon driven closed 5-bar mechanism has been realized to be used as a two-degree-of-freedom planar Haptic Interface, shown in figure 1. As the implemented tendon drive introduces a linear constraint between the joint angles of the mechanism, it gives raise to a more complicated study of its dynamic behavior.

Hamiltonian mechanics represents a powerful method for solving the dynamics equations of a lagrangian system with constraints (Arnold, 1989). The hamiltonian formulation allows to interpret the equations of the dynamics of the system as a flow in the co-tangent space to the configuration manifold. In this conceptual framework, fundamental concepts such as conservation of the generalized momenta or zero-dynamics of the system can be easily derived. This paper presents an example of application of these concepts to the dynamic modeling of an elastic transmission system intrinsically coupled with a closed 5-bar mechanism.

2 System Description

Two rotary actuators drive a closed 5-bar linkage by two pairs of opposed tendons realized through steel cables. The actuators are located apart from the linkages of the mechanism. Figure 3 shows the kinematics of the routing of the tendon drive for the first motor.

The starting terminal of a single tendon connects to the pulley mounted on the motor shaft, whilst its end terminal is attached to the grounded base link. Guide pulleys of different radii route orderly the tendons clockwise or counter-clockwise over circular primitives centered at the
joint axes. In the developed design all guide-pulleys are mounted on ball bearings, but the last one is bolted to the base link. This mechanical design allowed to reduce sliding friction between cables and pulley races. Clearly the routing of figure 3 is completed for each motor by a second tendon that is routed in an opposite way on the joint pulleys, in order to realize a pre-tensioned bi-directional tendon drive.

![Image](image)

**Figure 1.** The real system.

![Graph](graph)

**Figure 2.** Left side: EF trajectories with base joints drive. Right side: EF trajectories with proposed tendon drive.

The tendon drive couples linearly the angular displacements of all the joints to the motor displacements. By actuating one motor, while the other one is locked, the mechanism motion is kinematically constrained by the tendon drive. The end effector trajectories, placed at $P_b$, traced with one degree-of-freedom locked, depend on the values of guide pulleys radii $r_{pt}$. Such radii
can be considered as free kinematic parameters, suitable for modifying the kinematic performance of the closed 5-bar linkage. The radii of pulleys in system of figure 1 have been chosen to optimize a kinematic Global Isotropy Index (GII), (Stocco, 1998), over the assigned workspace. A value of GII, given by the ratio of the minimum semi-axis of all the kinematics manipulability ellipses in the given workspace to the maximum one, up to 0.8758 has been achieved (Frisoli, 1999). In order to realize a symmetric force behavior of the mechanism, the radii of the of pulleys for the transmissions of motor 1 and 2 satisfy the following relation \(^1\):

\[
r_{p1}^1 = r_{p3}^2 \quad r_{p2}^1 = r_{p4}^2 \quad r_{p3}^1 = r_{p1}^2 \quad r_{p4}^1 = r_{p2}^2 \quad r_{p5}^1 = r_{p5}^2
\]  

(1)

Figure 2 shows how the EF trajectories, obtained with the proposed tendon drive, differ from the ones obtained with the direct drive of base joints. It is evident the motion decoupling of the manipulator into two orthogonal directions.

\[\text{Figure 3. Kinematic representation of the system with relative notation.}\]

3 System Kinematics

The closed 5-bar linkage is composed of the two serial sub-chains \(P_1P_2P_3\) and \(P_3P_4P_5\) (see fig. 3), that connect the point \(P_5\) to the base link. The posture of the closed loop 5-bar linkage is

\(^1\) Herein and after the apex index indicates the transmission number
constrained by a vector loop equation, that in differential form can be simply expressed as:

$$\mathbf{p}_5 = J_{12} \begin{bmatrix} \dot{q}^1 \\ \dot{q}^2 \end{bmatrix} = J_{34} \begin{bmatrix} \dot{q}^3 \\ \dot{q}^4 \end{bmatrix}$$  \hspace{1cm} (2)$$

where the jacobians are given by:

$$J_{12} = \begin{pmatrix} -l_{12}s_1 & -l_{25}s_{12} \\ +l_{12}c_1 & +l_{25}c_{12} \end{pmatrix}, \quad J_{34} = \begin{pmatrix} -l_{12}s_3 & -l_{25}s_{34} \\ +l_{12}c_3 & +l_{25}c_{34} \end{pmatrix}$$  \hspace{1cm} (3)$$

3.1 Kinematic of the Tendon Drive

The kinematics of the tendon driven system depends both on the stretching of the tendon branches between pulleys $d_{i,j}^k$ and on the motor displacements $q_{in}^k$ (for reference see figure 3). The effect of the tendon stretching on the motion of the end effector with locked motors can be visualized as the effect of a strain induced by temperature. If the tendon branches are stretched of $\Delta d = \epsilon_1 d$, since geometrically the length of the tangent line between the circumferences of two pulleys remains constant, the mechanism can compensate the $\Delta d$ only by changing its posture towards a new one, with a greater length of the cable routed over pulleys.

![Figure 4](image-url)  

Figure 4. The observer placed on link 1, which can observe the displacement of the tendon branch, sees the idle pulleys rotating with opposite velocity $-q^1$.

The rotation of pulleys $q_{p,i}^k$ ($k$ assuming values 1, 2 according to the transmission) is considered positive when the rotation is concord with the positive tendon displacements. The pulleys angles $q_{p,i}^k$ are measured assuming as reference the angles determined by the system in the zero
configuration, after that the tendon routing has been pre-tensioned. The strain $\epsilon_0$ induced by the
pre-tensioning is modeled as a length increment of $\epsilon_0 d_{i,j}^k$.

The direction of $d_{i,j}^k$ is assumed positive when the cable is stretched. The terminal of each
tendon branch is considered attached to the pulley groove on which is routed. Each tendon branch
is modeled as a linear spring between two pulleys, so that the factors $d_{i,j}^k$ represent only the tendon
displacement stored as elastic energy. As the routing for each motor is realized by a pair of two
pre-tensioned tendons, routed in a opposite way, with linear stiffness $k_1$, we will equivalently
consider each motor connected to a single tendon only, but with linear stiffness equal to $2k_1$.

The winding/unwinding of the tendon over pulleys can be easily studied for each tendon
branch in a frame fixed to the link containing the branch under study. If we assume that the
orientation of the idle pulley is fixed in the space (see figure 4), with reference instance to
pulley $q_{i,j}^1$, in a system fixed to the pulley, the rotation of link 1 is observed as a opposite rotation
of $q_1$, while link 2 is observed moving of an angle of $q_1 + q_2^2$. With such conventions, it is easy
to write the expression of the length of each tendon branch:

\[
\begin{align*}
  d_{m,1,i1}^1 &= \epsilon_0 l_{m,1,i1}^1 + rq_{m,i1}^1 - rq_{1,1}^1 \\
  d_{1,1,i1}^1 &= \epsilon_0 l_{1,1,i1}^1 + r_q1^1 - r_p^1 q_{1,1}^1 \\
  d_{p,1,i2}^1 &= \epsilon_0 l_{p,1,i2}^1 + r_p^1 (q_{1,1}^1 - r_q1^1) - r_{p,1}^1 (q_{1,1}^1 - r_q1^1) \\
  d_{p,2,i2}^1 &= \epsilon_0 l_{p,2,i2}^1 + r_p^2 (q_{1,2}^1 - r_q1^1) - r_{p,2}^1 (q_{1,2}^1 - r_q1^1) \\
  d_{p,2,i2}^1 &= \epsilon_0 l_{p,2,i2}^1 + r_p^2 (q_{1,2}^1 - r_q1^1) - r_{p,2}^1 (q_{1,2}^1 - r_q1^1) \\
  d_{p,3,i3}^1 &= \epsilon_0 l_{p,3,i3}^1 + r_p^3 (q_{1,3}^1 - r_q1^1) - r_{p,3}^1 (q_{1,3}^1 - r_q1^1) \\
  d_{p,4,i3}^1 &= \epsilon_0 l_{p,4,i3}^1 + r_p^4 (q_{1,4}^1 - r_q1^1) - r_{p,4}^1 (q_{1,4}^1 - r_q1^1) \\
  d_{p,5,i4}^1 &= \epsilon_0 l_{p,5,i4}^1 + r_p^5 (q_{1,5}^1 - r_q1^1) - r_{p,5}^1 (q_{1,5}^1 - r_q1^1) \\
  d_{p,6,i5}^1 &= \epsilon_0 l_{p,6,i5}^1 + r_p^6 (q_{1,6}^1 - r_q1^1) - r_{p,6}^1 (q_{1,6}^1 - r_q1^1)
\end{align*}
\]

(4)

By summing all the previous equations, finally we obtain

\[
  rq_{m,i}^1 = \sum_i (d_{i,i+1}^1 - \epsilon_0 l_{i,i+1}^1) - (r_{p,1}^1 + r_p^1) q_1 - (r_{p,2}^1 + r_p^1) q_1^2 + (r_{p,3}^1 + r_p^1) q_1^3 + (-r_{p,4}^1 + r_p^1) q_4
\]

(5)

and with the assumption that $d^1 = \sum_i d_{i,i+1}^1$:

\[
  rq_{m,i}^1 = \sum_i (\epsilon_0 l_{i,i+1}^1) = n_{11} q_1^1 + n_{12} q_1^2 + n_{13} q_1^3 + n_{14} q_1^4 + d^1
\]

(6)

The equivalent equation of (6) for the second transmission can be found by respectively
replacing $\{q_1^1, q_2^2, q_3^3, q_4^4\}$ with $\{-q_3^3, -q_4^4, -q_3^3, -q_4^4\}$ \footnote{The sign minus takes into account that the angle conventions chosen for $q_1^1, q_2^2$ and $q_3^3, q_4^4$ are opposite in sign.}, and with the assumption that $d^2 = \sum_i d_{i,i+1}^2$:

\[
  rq_{m,i}^2 = \sum_i (\epsilon_0 l_{i,i+1}^2) = -n_{13} q_1^1 - n_{14} q_1^2 - n_{11} q_3^3 - n_{12} q_4^4 + d^2
\]

(7)
If the rotational dynamics of the idle pulleys is considered negligible, we can assume as configuration variables for the tendon drive the total lengths of both transmissions \( \{d^1, d^2\} \). Define the following state vectors:

\[
\mathbf{q}^T = \begin{bmatrix} q^1 & q^2 & q^3 & q^4 & d^1 & d^2 \end{bmatrix} = [\mathbf{q}^T \ d^T] \quad \tilde{\mathbf{q}}^T = [d^1_m \ d^2_m \ d^1 \ d^2] = [\mathbf{d}_{\tilde{m}}^T \ d^T]
\]

with \( d^i_m = r q^i_m \). Vector \( \mathbf{q} \) belongs to the configuration manifold of the mechanism \( \mathcal{M} \), while vector \( \tilde{\mathbf{q}} \) belongs to the constrained configuration manifold \( \mathcal{M}_0 \), under the constraint given by the loop equation (2). We can find the mapping from the tangent space of \( \mathcal{M} \) to the tangent space of \( \mathcal{M}_0 \), proceeding as follows.

Derive equations (6) and (7) with respect to the time, to obtain a differential equality, that with the aid of (8), can be arranged in a matrix form as:

\[
\mathbf{d}_m = \begin{pmatrix} n_{11} & n_{12} & n_{13} & n_{14} \\ -n_{13} & -n_{14} & -n_{11} & -n_{12} \end{pmatrix} \mathbf{q} + \dot{\mathbf{d}} = \tilde{\mathbf{N}} \mathbf{q} + \dot{\mathbf{d}}
\]

So that

\[
\dot{\mathbf{q}} = \begin{pmatrix} \mathbf{d}_m^T \\ \dot{\mathbf{d}} \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{N}} I_2 \\ 0_2 I_2 \end{pmatrix} \begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{d}} \end{pmatrix} = \mathbf{N} \dot{\mathbf{q}}
\]

By (2), the constraint on joint velocities can be expressed as:

\[
(J_{12} - J_{24} 0_2) \dot{\mathbf{q}} = A^T(\mathbf{q}) \dot{\mathbf{q}} = 0
\]

We can cast together equations (10) and (11), to obtain the following differential relation:

\[
\begin{pmatrix} \mathbf{N} \\ A^T \end{pmatrix} \dot{\mathbf{q}} + \mathbf{T} \dot{\mathbf{q}} = \begin{pmatrix} \dot{\mathbf{q}} \\ 0 \end{pmatrix}
\]

where matrix \( T \) is an invertible square matrix. By inverting equation (12), it easy to show that there exists a matrix \( S \) of dimension \( 6 \times 4 \) such that \( \dot{\mathbf{q}} = S(\mathbf{q}) \dot{\mathbf{q}} \), which clearly for the conservation of the energy satisfies \( \mathcal{N} S = I_4 \). The matrix \( S \) maps the vectors \( \dot{\mathbf{q}} \) of the tangent space to the constrained configuration manifold \( \mathcal{M}_0 \) into the tangent space of the configuration manifold \( \mathcal{M} \). As the vectors \( \dot{\mathbf{q}} \) must satisfy the constraint equation, this implies that \( \mathcal{I} \mathcal{M}(S) = \mathcal{Ker}(A^T) \).

A basis for the \( \mathcal{Ker}(A^T) \) can be easily written in matrix form as:

\[
\mathcal{Ker}(A^T) = K = \begin{pmatrix} I_2 & I_2 & J_{34}^{-1} J_{12} & J_{34}^{-1} J_{12} \\ J_{34}^{-1} J_{12} & J_{34}^{-1} J_{12} & 0_2 & 0_2 \end{pmatrix} (k_1 \ldots k_4) \rightarrow A^T K = 0
\]

It can be easily shown that the distribution \( \Delta_K \) defined by the column vectors of \( K \) is involutive, as the Lie bracket \([k_i, k_j] = 0, \forall i, j\). This result can be justified in reason of the planar kinematics of the mechanism, since the Lie group of translations in the plane is closed under the Lie bracket product.
4 Hamiltonian Formulation of the System Dynamics

If we define in the co-tangent spaces to $\mathcal{M}$ and $\mathcal{M}_0$, the generalized momenta as:

$$
\mathbf{p}^T = \begin{bmatrix} p_{1,1} & p_{1,2} & p_{1,3} & p_{1,4} & p_{1,5} & p_{1,6} \\ p_{2,1} & p_{2,2} & p_{2,3} & p_{2,4} & p_{2,5} & p_{2,6} \end{bmatrix} = \begin{bmatrix} \mathbf{p}_1^T \\ \mathbf{p}_2^T \end{bmatrix}
$$

$$
\tilde{\mathbf{p}}^T = \begin{bmatrix} p_{1,1} & p_{1,2} & p_{1,3} & p_{1,4} & p_{1,5} & p_{1,6} \\ p_{2,1} & p_{2,2} & p_{2,3} & p_{2,4} & p_{2,5} & p_{2,6} \end{bmatrix} = \begin{bmatrix} \mathbf{p}_m^T \\ \tilde{\mathbf{p}}_l^T \end{bmatrix}
$$

(14)

where the generalized momentum $\tilde{\mathbf{p}}$ is given by:

$$
\tilde{\mathbf{p}} = \begin{bmatrix} \mathbf{p}_m^T \\ \tilde{\mathbf{p}}_l^T \end{bmatrix}
$$

(15)

The kinetic energy of the system can be expressed as

$$
\mathcal{T} = \frac{1}{2} \dot{\mathbf{q}}^T \dot{\mathbf{q}} + \frac{1}{2} \mathbf{I}_m \dot{\mathbf{q}} + \frac{1}{2} \mathbf{q}^T (\mathbf{S}^T \mathbf{M} \mathbf{S} + \mathbf{I}_m) \dot{\mathbf{q}} - \frac{1}{2} \mathbf{q}^T \mathbf{M} \ddot{\mathbf{q}}
$$

(16)

with $\mathbf{I}_m = \text{diag}(I_{m1}, I_{m2}, 0, 0)$ and

$$
\mathbf{M} = \begin{bmatrix}
\begin{array}{cccccc}
-\frac{1}{2} \frac{1}{2} \\
0 & 0 & m^{32} (1 + 2 \gamma_1 c_2) & m^{34} (1 + \gamma_3 c_4) & 0 & \\
0 & 0 & m^{32} (1 + \gamma_1 c_2) & m^{34} (1 + \gamma_3 c_4) & 0 & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
0 & 0 & m^{32} (1 + \gamma_1 c_2) & m^{34} (1 + \gamma_3 c_4) & 0 & \\
0 & 0 & m^{32} (1 + \gamma_1 c_2) & m^{34} (1 + \gamma_3 c_4) & 0 & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
0 & 0 & m^{11} (1 + 2 \gamma_1 c_2) & m^{12} (1 + \gamma_1 c_2) & 0 & \\
0 & 0 & m^{11} (1 + 2 \gamma_1 c_2) & m^{12} (1 + \gamma_1 c_2) & 0 & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
0 & 0 & m^{11} (1 + 2 \gamma_1 c_2) & m^{12} (1 + \gamma_1 c_2) & 0 &
\end{array}
\end{bmatrix}
$$

(17)

where $c_i = \cos(q^i)$ and the other parameters can be found by the dynamic equations of a planar 2R (see Sciacicco, 1995). We can model the transmission as a linear spring of constant $k = 2k_1 L$, with $L$ the total length of one single tendon transmission and $k_1$ the linear stiffness of the tendon. Then, since the only storage of potential energy is represented by the tendon drive, the potential energy can be simply expressed as a function of the displacement vector $\mathbf{d}$:

$$
\mathcal{U} = \mathcal{U}_0 + \frac{1}{2} \mathbf{q}^T K \mathbf{q}
$$

where the term $\mathcal{U}_0$ is due to the pretensioning. So that finally, by denoting $\tilde{\mathbf{M}} = \mathbf{M}^{-1}$, the Hamiltonian (Arnold, 1989) of the system is given by:

$$
\mathcal{H} = \mathcal{K} + \mathcal{U} = \frac{1}{2} \tilde{\mathbf{p}}^T \tilde{\mathbf{M}} \tilde{\mathbf{q}} + \frac{1}{2} \mathbf{q}^T K \mathbf{q}
$$

(19)

Then following Van Der Schaft (1994), the equations of the system constrained dynamics can be written as:

$$
\begin{bmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} 0_{6,6} \\ -S^T(q) \end{bmatrix} \begin{bmatrix} S(q) \\ -T[S_i, S_j](q)i,j \end{bmatrix} + \begin{bmatrix} 0_{6,6} \\ (B_c) \end{bmatrix} \mathbf{u}
$$

(20)

$^3$ For the principle of virtual works $\mathbf{p}$ satisfies the equation $\mathbf{p} = S^T q$.

$^4$ $\mathcal{L}$ is the Lagrangian of the system.
with $\mathbf{p} = N^T \mathbf{p}$, where $X = (p^T S, S^T)_{1, i, j}$ expresses the coupling between the configuration variables due to the parallel kinematics of the mechanism. As $Im(S) = Ker(A^T)$, the columns of matrix $S$ are linearly dependent from the column of matrix $K$. Since the distribution $\Delta K$ is involutive, this implies that matrix $X$ is constantly equal to zero.

If we consider that the motors are applying an input torque $\tau_{m, i}$, a control input vector can be defined as:

$$\mathbf{u} = \left[ \tau_{m,1}, \tau_{m,2} \right]^T = \left[ T_{m,1}, T_{m,2} \right]^T$$  \hspace{1cm} (21)

In the constrained configuration space the control input $\mathbf{u}$ is related to the derivative of the momenta $\dot{\mathbf{p}}$ by the expression of $B_c$:

$$B_c = \begin{pmatrix} I_2 \\ 0_2 \end{pmatrix}$$  \hspace{1cm} (22)

If we compute the partial derivatives of the Hamiltonian expression (19), we find:

$$\begin{align*}
\dot{p}_i &= \frac{\partial H}{\partial \dot{p}_i} = \dot{M}^{ij} \dot{p}_j \\
-\dot{q}_i &= \frac{\partial H}{\partial \dot{q}_i} = \frac{1}{2} \frac{\partial \dot{M}^{jk}}{\partial \dot{q}_i} \dot{p}_j \dot{p}_k + k_{ij} q^j = c_{ij}^{kl} \dot{p}_k \dot{p}_l + k_{ij} q^j
\end{align*}$$  \hspace{1cm} (23)

where indexes and pedices are chosen according to the tensor notation. After having computed the values of terms $c_{ij}^{kl}$, we find that equation (20) can be rewritten the sum of a linear and non-linear term:

$$\begin{pmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{p}} \end{pmatrix} = \begin{pmatrix} 0_{6,6} & SM(q) \\ -S^T(q)K & 0_{4,4} \end{pmatrix} \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{r} \end{pmatrix} + \begin{pmatrix} 0 \\ B_c \end{pmatrix} T_m$$  \hspace{1cm} (24)

with

$$\begin{pmatrix} \mathbf{r} \\ \mathbf{r} \end{pmatrix} = \begin{pmatrix} a_{11} \dot{p}_{1,1}^2 + a_{12} \dot{p}_{1,1} \dot{p}_{1,2} + a_{22} \dot{p}_{1,2}^2 \\ b_{11} \dot{p}_{1,1}^2 + b_{12} \dot{p}_{1,1} \dot{p}_{1,2} + b_{22} \dot{p}_{1,2}^2 \\ -a_{11} \dot{p}_{1,1}^2 - a_{12} \dot{p}_{1,1} \dot{p}_{1,2} - a_{22} \dot{p}_{1,2}^2 \\ -b_{11} \dot{p}_{1,1}^2 - b_{12} \dot{p}_{1,1} \dot{p}_{1,2} - b_{22} \dot{p}_{1,2}^2 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ -r_1 \\ -r_2 \end{pmatrix}$$  \hspace{1cm} (25)

and coefficients $a_{ij}, b_{ij}$ depending on the mechanism configuration. So the generalized momenta satisfy the following relations:

$$\begin{align*}
\dot{p}_{m, i} &= T_{m, i} + r_i (\dot{p}_{1,1}, \dot{p}_{1,2}) \\
\dot{q}_{i} &= -kd_i^2 - r_i (\dot{p}_{1,1}, \dot{p}_{1,2}) \hspace{1cm} i = 1, 2
\end{align*}$$  \hspace{1cm} (26)

As the momenta $p_{m, i}$ do not depend on $q^i$, the first part of the system of equations (24) can be replaced with the expression of $\dot{\mathbf{q}}$ as a function of $\mathbf{p}$:

$$\dot{\mathbf{q}} = \ddot{M}(q) \ddot{p}$$

$$\ddot{M} = \begin{pmatrix} \frac{1}{M_{mm}} & 0 & 0 \\ 0 & \frac{1}{M_{pp}} & 0 \\ 0 & 0 & \frac{1}{M_{mm}} \end{pmatrix}$$  \hspace{1cm} (27)
where the terms \( M_m \) and \( M_l \) can be interpreted respectively as the equivalent masses of the motors and of the links reduced to the motor displacement \( d_m^i \), and:

\[
\frac{1}{M_p} = \frac{1}{M_{m|i}} = \frac{1}{M_m} + \frac{1}{M_l}
\]  

(28)

As also \( \frac{1}{M_{m|i}} \) can be assumed negligible in most of the workspace, in reason of the highly isotropic behavior of the mechanism, we can write (27) for \( i = 1, 2 \) as:

\[
\begin{align*}
\dot{d}_m^i &= \frac{1}{M_m} (p_{m,i} + \tilde{p}_l,i) \\
\dot{d}^i &= \frac{1}{M_m} p_{m,i} + \frac{1}{M_p} \tilde{p}_l,i
\end{align*}
\]  

(29)

If we consider as output the motor displacement \( d_m^i \), equations (29) and (26) can be split into the linearizable input-output dynamics and the zero output dynamics (Kwatra, 2000) of the system, by rewriting them as a function of the following new configuration variables:

\[
\begin{align*}
z_1^i &= d_m^i \\
z_2^i &= \frac{p_{m,i}}{M_m} + \frac{\tilde{p}_l,i}{M_m} \\
\xi_1^i &= d^i \\
\xi_2^i &= \tilde{p}_l,i
\end{align*}
\]  

(30)

with \( i = 1, 2 \) and

\[
v^i = \alpha^i(\xi) + \rho^i(\xi)T_m \\
\alpha^i = -k\xi_1^i \\
\rho^i = \frac{1}{M_m}
\]  

(31)

The non-linear zero-dynamics is given by:

\[
\begin{align*}
\dot{\xi}_1^i &= z_2^i + \frac{1}{M_p} \frac{1}{M_m} \\
\dot{\xi}_2^i &= -k\xi_1^i - r_i(\xi_1^i, \xi_2^i)
\end{align*}
\]  

(32)

It easy to show that if \( z_1^i = 0 \) and \( v = 0 \) (this implies an input torque of \( \tau_{m,i} = rT_{m,i} = -\alpha^i/\rho^i \)), then the dynamic equations (29) and (26) reduce to (32). Then there exists a dynamics of the system, described by (32), that does not produce any displacement \( d_m^i \) as output. If we linearize the equation (32) in the neighborhood of \( z = 0 \), by computing the associated jacobian matrix \( J \), we have an estimation of the modes of the system associated with the zero dynamics. In particular the eigenvalues of the jacobian \( J \) are found to be equal to:

\[
J = \begin{pmatrix} 0 & \frac{1}{M_p} - \frac{1}{M_m} \\ -k & 0 \end{pmatrix} \\
\omega_{1/2} = \pm \sqrt{\frac{k}{M_l}}
\]  

(33)
which corresponds to 71 Hz. The clear physical interpretation is that during the zero-dynamics, the links and tendons are moving with locked motors. If we linearize the equations (29) by using a Jacobian approximation, we can compute the LTI transfer function between \( T_{m,i} \) and \( d^i_m \) as:

\[
d^i_m = \frac{(M_1s^2 + k)}{s^2(M_mM_1s^2 + k(M_m + M_1))} T_{m,i}
\]

(34)

where the double pole of the origin corresponds to the eigenvalues associated to the linear dynamics (30). This can be easily recognized to be the dynamics of the equivalent 1-DOF system shown in figure 5.

5 Conclusions

The paper has presented an Hamiltonian formulation of the constrained dynamics of a closed 5-bar mechanism. The derivation of the dynamic equations has led to the identification of an equivalent 1-DOF model for each motor.

Hamiltonian mechanics represents a powerful theory for solving the dynamic equations of constrained parallel mechanisms.

References


